The Weibull Distribution to Represent Turbulent Transfer

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The most appropriate and natural basis for investigations and description of the phenomena of momentum, heat, and mass transfer is the boundary layer theory. However, solutions derived from this theory that can be applied to practical cases are often too complicated or even infeasible. Purposeful simplifications of the transfer mechanism cause a simplification of a mathematical description of transfer phenomena and result in the formation of transfer models (Bird et al., 1960; Krishna and Taylor, 1986).

Concept of the Turbulent Flux Density Distribution

The existing transfer models enable us to describe steady-state momentum, heat, and/or mass transport in the direction normal to the interface. Such transport occurs in the wall region of a flat and developed boundary layer (Bradshaw, 1975). The turbulent stress τ_t or heat and/or mass flux J_t in that region define the following equations written in dimensionless form as

$$\tau_t^+ = 1 - \frac{du^+}{dy^+}, \tag{1}$$

$$J_t^+ = 1 - \frac{1}{\sigma}\frac{d\psi^+}{dy^+}. \tag{2}$$

From Eqs. 1 and 2 it follows that the dimensionless turbulent stress (or turbulent flux) has a property of some probability function. Thus, on the analogy of probability density a notion of dimensionless turbulent stress density F^+ and dimensionless turbulent heat and/or mass flux density F_J^+ was proposed (Rogacki, 1985)

$$F^+ \equiv \left|\frac{d\tau_t^+}{dy^+}\right| = \left|\frac{d^2u^+}{(dy^+)^2}\right|, \tag{3}$$

$$F_J^+ = \left|\frac{dJ_t^+}{dy^+}\right| = \frac{1}{\sigma}\left|\frac{d^2\psi^+}{(dy^+)^2}\right|. \tag{4}$$

The effect of σ on the position and shape of the distribution of turbulent flux density is analogous to that imposed by the Pr/Sc number on the structure and mutual position of hydrodynamic, thermal, and concentration boundary layers.

Formulation of the Model

The proposed model postulates that the turbulent flux density of momentum, heat, and/or mass has the form of the Weibull distribution (Feller, 1966):

$$F_m^+ = \frac{n}{(as)}(\eta^+)^{n-1}\exp\left[-\frac{(\eta^+)^n}{(as)}\right]. \tag{5}$$

Since the model describes the phenomena of momentum, heat, and mass transport, the variable Π has been defined as a dimensionless velocity, temperature, or concentration, as given in Table 1. The dimensionless dependent variable Π^+ and distance from the interface is defined as

$$\Pi^+ \equiv \frac{\Pi}{N_0} \quad \text{and} \quad \eta^+ \equiv \eta. \tag{6}$$

The expression $1/(as)$ is an analog of the Pr/Sc number as it contains the molecular diffusivity a.

Using the relation between the turbulent flux density and the second derivative of the independent variable, Eq. 4, we have

$$F_m^+ = (as)\left|\frac{d^2\Pi^+}{d(\eta^+)^2}\right|. \tag{7}$$

Substituting Eq. 7 into Eq. 5 and taking advantage of the definitions in Eq. 6, one obtains

Table 1. Definitions of Dimensionless Variables and Modulus ($a\ s$)

Transfer of	Π	$(a\ s)$
Momentum	$\frac{u - u_0}{u_\infty - u_0}$	(νs)
Heat	$\Theta \equiv \frac{T - T_0}{T_\infty - T_0}$	$\left(\frac{\lambda}{\rho c_p} s\right)$
Mass	$\Phi \equiv \frac{C - C_0}{C_\infty - C_0}$	$(\mathcal{D}_{AB} s)$

$$\frac{d^2\Pi}{d\eta^2} = N_0 \frac{n}{(as)^2} \eta^{n-1} \exp\left[-\frac{\eta^n}{(as)}\right] \tag{8}$$

After integration of Eq. 8 we can find the profile of the generalized dependent variable as the quotient of the incomplete and complete gamma functions

$$\Pi = \frac{n}{\Gamma(1/n)(as)^{(1/n)}} \int_0^\eta \exp\left[-\frac{\eta^n}{(as)}\right] \quad d\eta = \frac{\Gamma\left(\frac{1}{n}, \frac{\eta^n}{(as)}\right)}{\Gamma(1/n)}. \tag{9}$$

The dimensionless flux of momentum, heat, and/or mass on the interface is equal to

$$N_{0*} = -(as)\frac{d\Pi}{d\eta}\bigg|_{\eta=0} = -\frac{n}{\Gamma(1/n)}(as)n - 1/n = -k\Delta\Pi. \tag{10}$$

Since the driving force in the dimensionless form is always equal to $\Delta\Pi = 1$,

$$k = \frac{n}{\Gamma(1/n)}(as)n - 1/n \tag{11}$$

Discussion

By Eq. 11 the transfer coefficient is proportional to the molecular diffusivity coefficient with varying power depending on the value of the parameter n. For $n = 1, 2, 3$, and $n \rightarrow \infty$ the exponent of this power is respectively zero (Reynolds analogy), 1/2 (penetration models, Danckwerts, 1951; Higbie, 1935), 2/3 (Ruckenstein, 1958, model), and 1 (film model); these are shown in Figure 1. In the following the latter three basic models are proved to be special cases of the model being proposed.

Assuming $n = 2$ and substituting

$$z^2 = \eta^2/(\mathcal{D}_{AB}s) \quad \text{and} \quad \eta \equiv \frac{y}{\delta} \tag{12}$$

into Eq. 9 we obtain the profile of dimensionless concentration described by the error function

$$\Phi = \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-z^2)dz = \text{erf}(z) \tag{13}$$

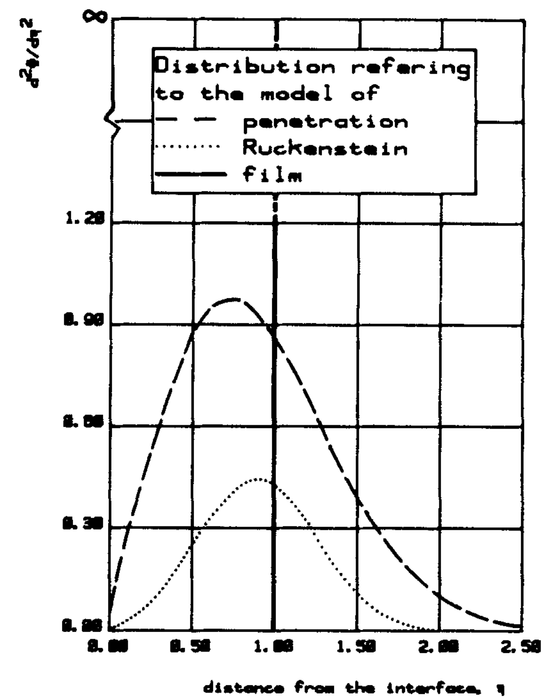

Figure 1. Distributions of $d^2\Phi/d\eta^2$ vs. distance from interface.

which is identical to that obtained from the penetration model. Comparison of the variable z defined in the penetration model as

$$z \equiv \frac{y}{2\sqrt{t_s \mathcal{D}_{AB}}} \tag{14}$$

with the definition in Eq. 12 enables us to determine the model parameter

$$t_s = \tfrac{1}{4}\delta^2 s \tag{15}$$

which corresponds to the exposure time occurring in Higbie's version of the penetration model.

The relation between the proposed model for $n = 3$ and the Ruckenstein model is revealed after comparing concentration profiles predicted by the proposed model

$$\Phi = \frac{\Gamma[1/3, \eta^3/(\mathcal{D}_{AB}s)]}{\Gamma(1/3)} \tag{16}$$

and Ruckenstein model (approximated to Pohlhausen's solution; Bird et al., 1960)

$$\Phi_R = \frac{\Gamma(1/3, \zeta\eta_R^3)}{\Gamma(1/3)}. \tag{17}$$

From the comparison it follows that for $n = 3$ both models predict the concentration profiles identically if

$$\frac{1}{(\mathcal{D}_{AB}s)} = Sc \quad \text{while} \quad \frac{\delta}{\delta_R} = \sqrt[3]{\frac{3!}{1.328}}. \tag{18}$$

When n tends to infinity in the proposed model, the turbulent flux density determined by the Weibull distribution takes a form of Dirac impulse. This impulse appears in the boundary layer at a constant distance $\eta = 1$ independent of the value of the product (as). This means that the effect of the diffusivity coefficient on the structure of the boundary layer stops, and that at the distance $y = \delta$ from the interface there is an infinitely abrupt increase of turbulence. Both these properties are characteristic for the film model.

Conclusions

In the proposed generalized model of momentum, heat, and mass transfer the concept of so-called turbulent flux density distribution was used. The model is based on the heuristic assumption that the turbulent flux density distribution can be described by the Weibull distribution. As an independent variable the distance from the interface has been assumed. The proposed model allows the mass transfer coefficients to be calculated as proportional to molecular diffusivity with varying power. Thus, through the proper change of the exponent over diffusivity we can describe the influence of diffusivity on transfer coefficients as predicted by other models (film, penetration, etc.). Since in the Weibull distribution the exponent need not be an integer, it can take the values occurring in other models, e g., in film-penetration models (Toor and Marcello, 1958), the Pinczewski-Sideman (1974) model, or in the boundary layer theory (Schlichting, 1979). Discussion of the properties of the proposed model shows that the concentration profiles obtained from the model overlap the profiles calculated from three fundamental transfer models.

Notation

a = generalized kinematic diffusivity coefficient, m^2/s
C = concentration, $kmol/m^3$
c_p = specific heat capacity, J/(kg · kg)
$\mathcal{D}_{AB}$ = molecular diffusivity coefficient, m^2/s
erf = error function
F = turbulent flux density
J = generalized flux of mass, heat, or momentum, $X/(m^2 \cdot s)$
k = transfer coefficient
n = exponent in Weibull distribution, parameter of generalized model
N_O = flux
s = parameter of Weibull distribution, parameter of generalized model, s/m^2
t_s = exposure time in penetration model, s
T = temperature, K
u = velocity, m/s
y = distance from interface, m
z = variable in penetration model

Greek letters

Γ = Gamma function
δ = characteristic length, m
$\Delta\Pi$ = generalized driving force
η = distance from interface
λ = thermal conductivity, J/(m · s · K)
ν = kinematic viscosity, m^2/s
Π = dependent variable in transfer model
ρ = density, kg/m^3
σ = generalized Prandtl/Schmidt number
τ = stress
θ = temperature profile
Φ = concentration profile
Ψ = generalized independent variable

Subscripts

J = connected with flux J
m = model
o = interface
R = Ruckenstein model
t = turbulent
∞ = bulk

Superscripts

$+$ = dimensionless variable

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Manuscript received Oct. 31, 1986, and revision received March 6, 1987.